ISRAEL JOURNAL OF MATHEMATICS, Vol. 53, No. 2, 1986

SEMI-GROUPS AND GRAPHS FOR SOFIC SYSTEMS

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ABSTRACT

Sofic systems are examined using defining semi-groups. The algebraic properties of the semi-groups, the graph-theoretic properties of the transition matrices, and the dynamic properties of the sofic systems are found to have many strong connections.

0. Introduction

Sofic systems were introduced into ergodic theory by B. Weiss [16] in 1972. He defined them using finite semi-groups and showed that they are the closure, among symbolic systems, of the subshifts of finite type under continuous factor maps. Subsequent work [4], [5], [6], and [7] investigated many of the basic properties of these systems and established that any sofic system is the 1-to-1 a.e. continuous image of a subshift of finite type.

Sofic systems had been studied under different names prior to 1972, most notably by engineers as finite state automata. In this paper, we will not discuss any of this work. See [10] and [13] for references.

Recently, there has been renewed interest in sofic systems. This has come about because of both practical [11] and dynamical [10], [12], [13] considerations.

Sofic systems differ from subshifts of finite type in that they have a finitary rather than finite memory. Subshifts of finite type support Markov measures and the set of Markov measures supported by a fixed subshift of finite type has a very rich structure [15]. The motivation for this work was to find a class of measures that plays the same role for sofic systems that the Markov measures play for subshifts of finite type. This would mean that the measures would have a finitary

† Partially supported by NSF Grants MCS-8120790-03 and MCS-8301324.
Received June 26, 1984

rather than finite conditional memory. To do this we returned to Weiss' original definition and found that the algebraic properties of the defining semi-group, the matrix (or graph) theoretic properties of his right and left transition matrices, and the dynamical properties of the sofic systems have surprisingly strong interrelation. These interrelations are discussed here, the measure theoretic questions will be addressed elsewhere.

Section 1 is devoted to background material.

In Section 2 we define the right and left transition matrices, principal components of the matrices, and recurrent semi-group elements. Then we establish some of the basic algebraic, graph theoretic, and dynamic connections. It is shown that any defining semi-group for a sofic system produces two canonical finite-to-one covers for the sofic system (i.e. subshifts of finite type with continuous, shift-commuting, onto maps from themselves to the sofic system). One is a left resolving cover and the other a right resolving cover.

Section 3 consists entirely of examples.

Section 4 goes more deeply into the connections between the algebraic, graph theoretic and dynamic properties. It is shown that all principal components of either transition matrix have the same number of symmetries. Then there is a duality that exists between the principal components of the left transition matrix and the principal components of the right transition matrix. This duality depends on the number of symmetries of these components. Finally we see that the canonical covers from Section 2 are s-to-1 a.e. where s is the number of symmetries. This allows us to construct 1-to-1 a.e. covers.

In Section 5 we discuss the minimal left and right resolving covers of a sofic system [6].

In Section 6 we show that any 1-to-1 a.e. left resolving cover and any 1-to-1 a.e. right resolving cover for a given sofic system can be obtained simultaneously as the canonical covers associated to a defining semi-group. We define the minimal (in terms of semi-group homomorphisms) semi-group for such a pair of covers. We show that there is a parallel between a certain type of semi-group homomorphism and factor maps between covers for a sofic system. This algebraic and dynamic parallel allows a very neat formulation.

In Section 7 we extend the results of Section 6, with the appropriate changes, to k-to-1 a.e. covers for sofic systems. These results are stated without proof.

1. Background

We begin with an n-point space $\{1,\dots,n\}$ with the discrete topology. The space $\{1,\dots,n\}^{Z}$ and the shift transformation. $(\sigma(x))_i = x_{i+1}$ make up the full

n-shift, (Σ_n, σ) or Σ_n. The product topology makes $\{1, \ldots, n\}^{\mathbf{Z}}$ a compact zero-dimensional topological space and σ a homeomorphism.

A subshift is a dynamical subsystem of a full n-shift. We require that the space be closed and shift-invariant, then restrict the shift to it. The topology is the subspace topology. For any subshift (X, σ), we define the alphabet L_x of X, to be the collection of symbols in $\{1, \ldots, n\}$ that occur in some point in X. We denote by $\mathcal{W}(X, n)$ the set of blocks of length n that occur in some point in X, and by $\mathcal{W}(X)$ or $\mathcal{W}$ the set $\bigcup_n \mathcal{W}(X, n)$. For a word $w \in \mathcal{W}(X)$ we define

$$p_x(w) \text{ or } p(w) = \{u \in \mathcal{W} : uw \in \mathcal{W}\} \quad \textit{and} \quad f_x(w) \text{ or } f(w) = \{u \in \mathcal{W} : wu \in \mathcal{W}\}.$$

These are the predecessor and successor sets of w. We say that a subshift is irreducible or transitive if it contains a point whose forward and backward orbits are dense. Such a point is called a *doubly transitive* point. The topological entropy of a subshift X is

$$h(X) = \lim_{n \to \infty} \frac{1}{n} \log B_n(X)$$

where $B_n(X)$ is the number of words of length n in $\mathcal{W}(X)$. This is always a well-defined, non-negative, finite number.

If X and Y are subshifts and $\phi : X \to Y$ is a continuous, onto, shift-commuting map then we say ϕ is a *factor map* or Y is a *factor* of X. Any factor map between two subshifts is a k-block map, for some k [9]. A $(2n+1)$*-block map* is a map $\phi : \mathcal{W}(X, 2n+1) \to L_Y$ so that

$$(\phi(x))_i = \phi([x_{i-n}, \ldots, x_{i+n}]) \qquad \text{for all } i \in \mathbf{Z}.$$

Here ϕ is used interchangeably as a map on the blocks or as a map on points. It is easy to see that if $\phi : X \to Y$ is a boundedly finite-to-one factor map then $h(X) = h(Y)$ [3].

A special class of the factor maps is the class of resolving maps. A 1-block factor map $\phi : X \to Y$ is called *right resolving* if for each $j \in L_Y$, $i \in \phi^{-1}(j)$ and $j' \in f(j) \cap L_Y$ there is at most one $i' \in f(i) \cap L_X$ such that $\phi(ii') = jj'$. A *left resolving* map is defined similarly but by using predecessors instead of successors. Notice that any resolving map is boundedly finite-to-one.

A special class of subshifts are the subshifts of finite type [14], [17], [1]. There are several ways to characterize these. The one we will use is to start with an $l \times l$ non-negative integer matrix A whose entries sum to n. The matrix A defines a directed graph with l vertices (we will denote the set of vertices by V_A) and $A(i, j)$ edges from vertex i to vertex j. Label each edge distinctly by an element of $\{1,\ldots,n\}$. Let $\Sigma_A \subseteq \{1,\ldots,n\}^{\mathbf{Z}}$ be the set of points x such that the edge labelled x_i ends at the vertex where the edge labelled x_{i+1} begins, for all $i \in \mathbf{Z}$. The subshift (Σ_A, σ) or simply Σ_A is the subshift of finite type (ssft) defined by A. We will think of a ssft as being defined by either a matrix or a directed graph and we will pass back and forth between these two points of view. For a word $w \in \mathcal{W}(\Sigma_A)$ examine $f(w)$. This set depends only on the last symbol of w, w_t. It consists of all words $u \in \mathcal{W}(\Sigma_A)$, that begin with a symbol whose edge begins at the vertex where w_t ends. Consequently, when dealing with subshifts of finite type it is customary to consider only successor sets of symbols and to consider them as made up of symbols. The same is true for predecessor sets. Any subshift with this property (that the predecessor/successor sets depend on only the first/last symbol) is a subshift of finite type. A ssft is irreducible if and only if the defining matrix or graph is irreducible, i.e. there is a path from any vertex of A to any other. The topological entropy of a ssft is $\log \lambda$, where λ is the spectral radius of the defining matrix. Unless otherwise stated all the ssft we deal with will be irreducible and have positive entropy.

Suppose $\phi : \Sigma_A \to X$ is a 1-block factor map from a ssft Σ_A to some subshift X. The map is boundedly finite-to-one if and only if there is no pair of words $i_0 i_1 \cdots i_{N-1} i_N, i_0 j_1 \cdots j_{N-1} i_N \in \mathcal{W}(\Sigma_A)$ with $i_l \neq j_l$, $1 \leqq l \leqq N-1$ and $\phi(i_0 i_1 \cdots i_{N-1} i_N) = \phi(i_0 j_1 \cdots j_{N-1} i_N)$ [3]. For such a finite-to-one ϕ and for $w = w_1 \cdots w_t \in \mathcal{W}(X)$ let

$$V_i(w) = \{v \in V_A : a_i \text{ ends at } v \text{ for some } a_1 \cdots a_t \in \phi^{-1}(w)\}, \qquad 1 \leqq i \leqq t$$

and

$$V_0(w) = \{v \in V_A : a_1 \text{ begins at } v \text{ for some } a_1 \cdots a_t \in \phi^{-1}(v)\}.$$

Then let

$$k = \inf_{w \in \mathcal{W}(x)} \left(\inf_i (\text{card } V_i(w)) \right).$$

Then the map ϕ is exactly k-to-1 on the doubly transitive points and we will say that it is k*-to-*1 *a.e.* [3]. A word where this infimum is attained is called a *magic word* for ϕ. If ϕ is right resolving k-to-1 a.e. and a word $w = w_1 \cdots w_t$ is magic then card $V_t(w) = k$. If, instead, ϕ is left resolving then card $V_0(w) = k$.

The sofic systems form another subcollection of the subshifts. This is the class we are primarily interested in. This collection contains the subshifts of finite type as a proper subset. We will follow Weiss [16] to define these. A *sofic system* S is defined (or described) by a *sofic pair* $(\mathcal{G}, \mathcal{A})$. Here $\mathcal{G}$ is a finite semi-group, with an absorbing element (or 0) such that $0 \cdot g = g \cdot 0 = 0$ for $g \in \mathscr{G}$, and $\mathscr{A} \subseteq \mathscr{G} \setminus \{0\}$ is a set of generators for $\mathscr{G}$. Think of $\mathscr{A} = \{1, \ldots, n\}$. A point x is in the subset S of $\mathscr{A}^{\mathbf{Z}} = \{1, \ldots, n\}^{\mathbf{Z}}$ if and only if every word occurring in x has a non-zero product in $\mathscr{G}$. The subshift (S, σ) or just S is the sofic system defined by $(\mathscr{G}, \mathscr{A})$. To see that every subshift of finite type is a sofic system, fix Σ_A. Let $\mathscr{A} = L_A$, $\mathscr{G} = \{0\} \cup (L_A)^2$. Identify $i \in L_A$ with the element (i, i) in $\mathscr{G}$ and define multiplication by:

$$(i, j)(k, l) = \begin{cases} (i, l) & \text{if } jk \in \mathscr{W}(\Sigma_A), \\ 0 & \text{otherwise.} \end{cases}$$

Then $(\mathscr{G}, \mathscr{A})$ describes Σ_A.

A sofic system is irreducible if and only if $(\mathscr{G}, \mathscr{A})$ is a sofic pair describing it and for every $g, h \in \mathscr{G} \setminus \{0\}$ there is a $k \in \mathscr{G}$ such that $gkh \neq 0$ [4]. We will see that the entropy of a sofic system is always the log of an algebraic number (Section 2). Unless otherwise stated we will always assume that a sofic system is irreducible and has positive entropy.

A word $w = w_1 \cdots w_t$ is in $\mathscr{W}(S)$ if and only if the product $g = w_1 \cdots w_t$ is not zero in $\mathscr{G}$. A word $a_1 \cdots a_m \in \mathscr{W}(S)$ is in $f(w)$ if and only if the product $ga_1 \cdots a_m = w_1 \cdots w_t a_1 \cdots a_m$ is non-zero in $\mathscr{G}$. In other words, $f(w)$ is completely determined by w's product in $\mathscr{G}$. This means there are only finitely many distinct follower sets in $\mathscr{W}(S)$. Sometimes we will think of $f(w)$ as a collection of words and sometimes as a collection of semi-group elements. All of this is true for the predecessor sets as well.

We will need a few facts from the theory of non-negative matrices [8]. A non-negative matrix A is *irreducible* if for each i, j there is an n such that $A^n(i, j) > 0$. An irreducible matrix has a *period*, $p = \gcd\{n : A^n(i, i) > 0, \text{all } i\}$.

PERRON–FROBENIUS THEOREM. *If A is a non-negative irreducible matrix with period p then*

(a) *There is a positive real eigenvalue λ with a corresponding strictly positive eigenvector,*

(b) *λ is a simple eigenvalue (i.e. a simple root of the characteristic polynomial),*

(c) *$\lambda\omega^i$ for $i = 0, \ldots, p-1$ are eigenvalues where ω is a primitive pth root of unity,*

(d) *for all eigenvalues μ with $\mu \neq \lambda w^i$, $i = 0, \ldots, p-1$, $|\mu| < \lambda$,*

(e) $\lim_{n\to\infty} A^{np}(i, i)/\lambda^{np} = r_i l_i$ *where r and l are right and left eigenvectors, $Ar = \lambda r$ and $lA = \lambda l$, with $lr = 1$.*

THEOREM. *Any non-negative square matrix A can, by applying a permutation to its rows and columns, be put into the form*

$$\left[\begin{array}{c|c|ccc|c} A_1 & 0 & \bullet & \bullet & \bullet & 0 \\ \hline A_{21} & A_2 & 0 & & & 0 \\ \hline \vdots & \vdots & \bullet & \bullet & \bullet & \vdots \\ & & & \bullet & \bullet & \\ & & & & \bullet & 0 \\ \hline A_{s1} & A_{s2} & & & & A_s \end{array}\right]$$

where each A_i is either irreducible or a one by one zero matrix. We will call $A_1, \ldots, A_s$ the components *of* A.

Notice that if A is irreducible, it has one component, itself. When A is integral and can be considered as a directed graph, this means that the vertices split into s disjoint subsets. The graph of any one of these subsets is irreducible or a single vertex without a loop (it is a component). One component may communicate (or exit) to another but if it does there will be no path from the new component back to the original one. There is at least one component that does not lead to any other component. This decomposition in conjunction with the Perron–Frobenius Theorem will be used repeatedly throughout the rest of the discussion.

2. Sofic graphs

For a sofic pair $(\mathcal{G}, \mathcal{A})$ we will follow Weiss [16] and define square non-negative integral (right and left transition) matrices R and L indexed by $\mathcal{G}\setminus\{0\}$ as follows:

$$R(g,h) = \text{card}\{a \in \mathcal{A} : ga = h\},$$

$$L(g,h) = \text{card}\{a \in \mathcal{A} : ag = h\}.$$

Since $\mathcal{G}$ is associative we have $RL = LR$. As with ssft we will sometimes think of R (or L) as a matrix and sometimes we will think of it as a directed graph whose vertices are the nonzero elements of $\mathcal{G}$ and whose edges are labelled by generators.

We will examine the structure of R and L. Let λ be the largest eigenvalue of R. If $g \in \mathcal{G}\setminus\{0\}$ and $\Theta_n(g)$ denotes the number of words in $\mathcal{A}^n$ whose product is g,

$$\overline{\lim}\, \frac{1}{n} \log \Theta_n(g) = \log \mu \qquad \text{for some } 0 \leqq \mu \leqq \lambda.$$

Moreover μ will be the largest of the spectral radii of the components of R that lead to g. This means that if g lies in a component of R that has λ for an eigenvalue, then

$$\overline{\lim}\, \frac{1}{n} \log \Theta_n(g) = \log \lambda.$$

It also means that R and L have the same largest eigenvalue and that the sofic system S defined by $(\mathcal{G}, \mathcal{A})$ has topological entropy $h(S) = \log \lambda$.

We will say that g is *recurrent* if $\overline{\lim}(1/n)\log \Theta_n(g) = \log \lambda$, that g is *transient* if

$$\overline{\lim}\, \frac{1}{n} \log \Theta_n(g) = \log \mu \qquad \text{for some } \mu < \lambda,$$

and we will denote by $\mathcal{G}_0$ the set of recurrent elements. A component of R (or L) will be said to be *principal* if its maximal eigenvalue is λ. Our aim is to see how these various components and elements are related. We will do this by making a series of observations.

In these observations, for notational simplicity, we will discuss the matrix R and ignore L. All of these observations are also true for the matrix L, when the appropriate changes are made. If any non-obvious changes in the statements are needed they will be mentioned (e.g. Observation 2).

OBSERVATION 1. If a component $\mathcal{C}$ of R has no exit then it is principal.

PROOF. Let g be an element of $\mathcal{C}$ and h be any recurrent element. Since S is irreducible there is an element k such that $gkh \neq 0$. Since $\mathcal{C}$ has no exits the element gkh is also in $\mathcal{C}$ and

$$\overline{\lim}\, \frac{1}{n} \log R^n(a, g) = \overline{\lim}\, \frac{1}{n} \log R^n(a, gkh) = \overline{\lim}\, \frac{1}{n} \log R^n(a, h)$$
$$= \log \lambda \quad \text{for some } a \in \mathcal{A}. \qquad \square$$

OBSERVATION 2. If $\mathcal{C}$ is an R component with no exit then the edge labelling of $\mathcal{C}$ defines a right resolving onto map $\phi : \Sigma_{\mathcal{C}} \to S$.

If $\mathscr{C}$ is an L component with no exit we must consider the ssft defined by $\mathscr{C}^t$ (the transpose). Then we have that the edge labelling of $\mathscr{C}^t$ defines a left resolving onto map $\phi : \Sigma_{\mathscr{C}^t} \to S$.

PROOF. The map ϕ is clearly a right resolving map into S. Let $a^1 \cdots a^n \in \mathscr{W}(S)$. Then for any $g \in \mathscr{C}$ there will be a k so that $(gk)h = (gk)a^1 \cdots a^n \neq 0$. Since $\mathscr{C}$ has no exit, $gka^1 \cdots a^n \in \mathscr{C}$. Then there is an $x \in \Sigma_{\mathscr{C}}$ where $\phi(x)$ contains $a^1 \cdots a^n$. □

COROLLARY [4]. *Every sofic system is the finite-to-one image of a ssft.*

OBSERVATION 3. If X is a proper subshift of S then $h(X) < h(S)$.

PROOF. Observation 2 shows that $\phi : \Sigma_{\mathscr{C}} \to S$ is an onto right resolving map. Since $\phi^{-1}(X)$ is a proper subshift of $\Sigma_{\mathscr{C}}$, and since any proper subshift of a ssft has strictly lower entropy,

$$h(X) = h(\phi^{-1}(X)) < h(\Sigma_{\mathscr{C}}) = h(S).$$ □

OBSERVATION 4. A component of R is principal if and only if it has no exits.

PROOF. We already know that a component with no exit is principal. Suppose $\mathscr{C}$, a component of R, has an exit. We will see that $\phi(\Sigma_{\mathscr{C}}) \subseteq S' \subsetneq S$ where S' is obtained by deleting a given block. This will mean that $\mathscr{C}$'s maximal eigenvalue is less than λ. Since $\mathscr{C}$ has an exit, there is an element $g \in \mathscr{C}$ and $a \in \mathscr{A}$ with $ga \neq 0$ and $ga \not\in \mathscr{C}$. Either there is an $h \in \mathscr{C}$ with $ha \neq 0$ or the symbol a doesn't occur in $\phi(\Sigma_{\mathscr{C}})$. If the second case holds, we are done. If the first case holds, there will be a block $a^1 \cdots a^n$ so that $haa^1 \cdots a^n = g$. Then $haa^1 \cdots a^n a$ and $gaa^1 \cdots a^n a$ are not elements of $\mathscr{C}$ and at least one is nonzero. We are back to the original situation, using $aa^1 \cdots a^n a$ instead of a. By continuing in this manner we can produce a word that occurs in S but not in $\phi(\Sigma_{\mathscr{C}})$. □

OBSERVATION 5. An element $g \in \mathscr{G}$ is recurrent if and only if it lies in a principal R component.

PROOF. If g lies in a principal component then

$$\overline{\lim}\, \frac{1}{n} \log R^n(a, g) = \log \lambda \qquad \text{for some } a \in \mathscr{A},$$

and g is recurrent. If g is recurrent and lies in a component $\mathscr{C}$, then

$$\overline{\lim}\, \frac{1}{n} \log R^n(a, g) = \log \lambda \qquad \text{for some } a \in \mathscr{A}.$$

This means that some path from a to g passes through a component whose maximal eigenvalue is λ. The previous observation shows that such components have no exits, so $\mathscr{C}$ has maximal eigenvalue λ. □

The gist of these observations is:

THEOREM 1. *For $\mathscr{C}$ a component of R, the following are equivalent*:
(i) *$\mathscr{C}$ is principal,*
(ii) *there is no exit from $\mathscr{C}$,*
(iii) *$\mathscr{C}$ contains a recurrent element,*
(iv) *every element of $\mathscr{C}$ is recurrent.*

These properties of principal components and recurrent elements will be used repeatedly in what follows. Also notice that $\mathscr{G}g\mathscr{G} \supseteq \mathscr{G}_0$ and that $\cap\, \mathscr{G}g\mathscr{G} = \mathscr{G}_0$ where the intersection is over all $g \in \mathscr{G}\setminus\{0\}$.

Next we will examine some of the relations that exist between the components of R. Suppose $g, h \in \mathscr{G}\setminus\{0\}$ and $gk = h$ for some k, then $lg \neq 0$ whenever $lh \neq 0$, or $p(h) \subseteq p(g)$. This leads to the following.

OBSERVATION 6. If g, h are in the same R component, then they have the same predecessors. □

If Γ is an irreducible subgraph of R, a *translation* is left multiplication of each element of Γ by an element $g \in \mathscr{G}$ so that $gh \neq 0$ for each $h \in \Gamma$ and so that $gh = gk$ implies $h = k$ for $h, k \in \Gamma$.

This means that $g\Gamma$ and Γ are identical subgraphs of R. In particular, Γ and $g\Gamma$ have the same maximal eigenvalue. Also, if $gh = h$ for some $h \in \Gamma$ then $gk = k$ for all $k \in \Gamma$. This is true since $k = hk'$ for some k' and $gk = ghk' = k$. The use of translations yields the following.

THEOREM 2. *All principal R components are identical* (*as irreducible labelled graphs*).

PROOF. Let $\mathscr{C}$, $\mathscr{C}'$ be two principal R components. We will translate $\mathscr{C}'$ onto $\mathscr{C}$. Let $g \in \mathscr{C}$ and $h \in \mathscr{C}'$. There is a $k \in \mathscr{G}$ so that $gkh \neq 0$. Since $\mathscr{C}$ is principal, $gkh \in \mathscr{C}$ and $gk(\mathscr{C}') \subseteq \mathscr{C}$. To see that this is a translation suppose $(gk)h = (gk)h'$ for $h, h' \in \mathscr{C}'$. By Theorem 1, gkh is in the same principal L component as h and there exists a $k' \in \mathscr{G}$ such that $(k'gk)h = h$. This means $h = h'$. Thus, $\mathscr{C}'$ is identical to a subgraph of $\mathscr{C}$. The Perron–Frobenius theorem then implies that they are identical. □

From now on we will denote by Σ_R the ssft defined by any principal R component, and by ϕ_R the right resolving onto map from Σ_R to S defined by the edge labelling. Similarly, Σ_L will denote the ssft defined by the transpose of any principal L component, and ϕ_L will denote the left resolving onto map from Σ_L to S. A pair (Σ_A, ϕ) where Σ_A is a ssft and ϕ is an onto map from Σ_A to S will be called a *cover*.

THEOREM 3. *Let S be a sofic system. Every pair* $(\mathcal{G}, \mathcal{A})$ *that describes S defines, in the above way, a unique right resolving cover* (Σ_R, ϕ_R) *and a unique left resolving cover* (Σ_L, ϕ_L) for S.

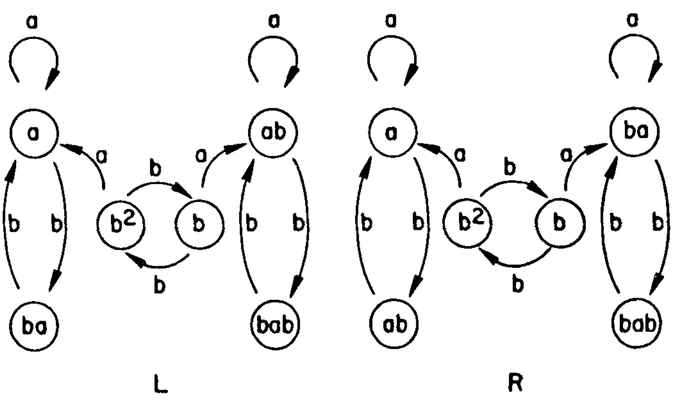
3. Examples

Here we present three examples. They illustrate the observations made in Section 2 as well as the ones that will follow in Section 4.

EXAMPLE 1. The *even* system of Weiss [16].
Let $\mathcal{A} = \{a, b\}$ and the relations be:

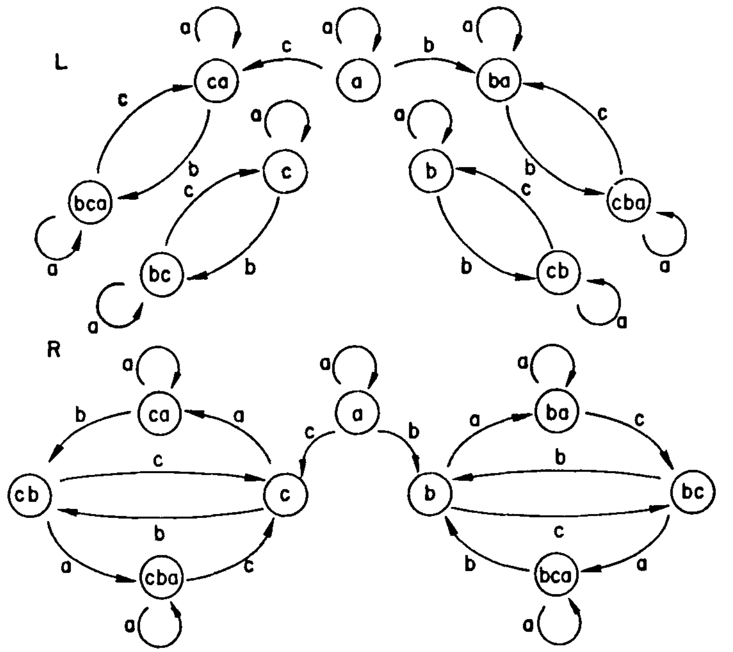
$$aba = 0, \quad a^2 = a, \quad b^3 = b, \quad ab^2 = b^2a = a.$$

Between any two occurrences of the symbol a we must see an even number of b's.



EXAMPLE 2. A before-after system. Let $\mathcal{A} = \{a, b, c\}$ and the relations be:

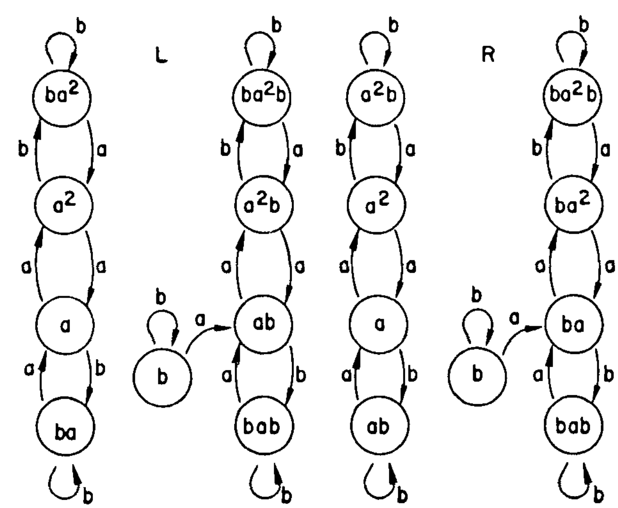
$$b^2 = c^2 = bab = cac = 0, \quad a^2 = a, \quad ab = b, \quad ac = c, \quad bcb = b, \quad cbc = c.$$



Here, if the last symbol before an unbroken string of a's is the symbol b then the first symbol after the string must be the symbol c, and vice-versa.

EXAMPLE 3. The two-shift. Let $\mathscr{A} = \{a, b\}$ and the relations be:

$$a^3 = a, \quad b^2 = b, \quad aba = a.$$



4. More on sofic graphs

A *symmetry* of an R component is a translation of the component onto itself. In example 3 of Section 3 each principal component has two symmetries, the identity symmetry and a flip symmetry. We will say that multiplication by two elements, g and h, give the same translation or symmetry of a component $\mathscr{C}$ if $gk = hk$ for each $k \in \mathscr{C}$.

OBSERVATION 7.

(a) If multiplication by g is a symmetry for $\mathscr{C}$ and $gh = h$ for some $h \in \mathscr{C}$, then it is the identity symmetry.

(b) For each principal component there is an identity symmetry.

(c) If g and h are in the same principal R component and there is a symmetry that takes g to h then g and h lie in the same principal L component.

(d) All principal R components have the same number of symmetries.

(e) The principal R and L components have the same number of symmetries.

(f) For each principal R component $\mathscr{C}$ and each principal L component $\mathscr{D}$ there is at least one element of $\mathscr{C}$ that lies in $\mathscr{D}$.

PROOF.

(a) This was proved in Section 2.

(b) If $\mathscr{C}$ is a principal R component and $g \in \mathscr{C}$ then g lies in some principal L component. There exists an element so that $hg = g$. By part (a) h gives the identity symmetry for $\mathscr{C}$.

(c) If g and h are in the same principal R component and there is a $k \in \mathscr{G}$ so that $kg = h$ then g and $kg = h$ are in the same principal L component.

(d) We know there is a translation that takes any principal component to any other (see proof of Theorem 2). Fix two such components $\mathscr{C}$ and $\mathscr{C}'$. There are elements g and g' such that $g\mathscr{C} = \mathscr{C}'$ and $g'\mathscr{C}' = \mathscr{C}$.

The map $h \rightarrow ghg'$ gives a bijection between the symmetries of $\mathscr{C}$ and the symmetries of $\mathscr{C}'$.

(e) It follows from (c) that the number of symmetries of a principal R component is the number of elements of $\mathscr{C}$ that lie in the same principal L component. Since the reverse is also true, the number of symmetries of the principal R and L components is the number of recurrent elements that lie in common R and L components.

(f) Fix a principal R component $\mathscr{C}$ and an element $g \in \mathscr{C}$. For any principal L component $\mathscr{D}$, choose an $h \in \mathscr{D}$. There is a $k \in \mathscr{G}$ so that $gkh \neq 0$. Notice $gkh \in \mathscr{C}$ and $gkh \in \mathscr{D}$. □

We may now define an equivalence relation $\sim$ on $\mathcal{G}_0$ by saying $g \sim h$ if and only if g and h are in the same R and L components. It follows from the previous observation that if s is the number of symmetries of a principal R component then s is the number of symmetries of a principal L component, and s is also the number of elements in each equivalence class. If r is the number of elements in a principal R component and l is the number of elements in a principal L component then r/s is the number of principal L components, l/s is the number of principal R components, and rl/s is the number of recurrent elements. We have a duality between the principal R components and the principal L components. Every principal R component contains s elements that lie in a given principal L component and vice versa. These s elements have the same predecessors and successors. Furthermore, the elements from a fixed principal R component play corresponding roles in their L components. In the examples of Section 3 we have $s = 1, 1$, and 2, respectively. Next we will examine how the duality relates to the dynamics of (Σ_R, ϕ_R), (Σ_L, ϕ_L) and S.

THEOREM 4. *If s is the number of symmetries of the principal components of $(\mathcal{G}, \mathcal{A})$ and (Σ_R, ϕ_R), (Σ_L, ϕ_L) are the covers then ϕ_R and ϕ_L are s-to-1 a.e.*

PROOF. First observe that if there are s symmetries, ϕ_R and ϕ_L are at least s-to-1 everywhere. Now suppose ϕ_R is k-to-1 a.e. We will show that there are at least k symmetries. Suppose $a^1 \cdots a^n$ is magic for ϕ_R, take $g = a^1 \cdots a^n$. Then there are elements $h^1, \ldots, h^k \in \mathcal{C}$ so that $h^1 g, \ldots, h^k g$ are distinct elements in $\mathcal{C}$. This means there are at least k symmetries. □

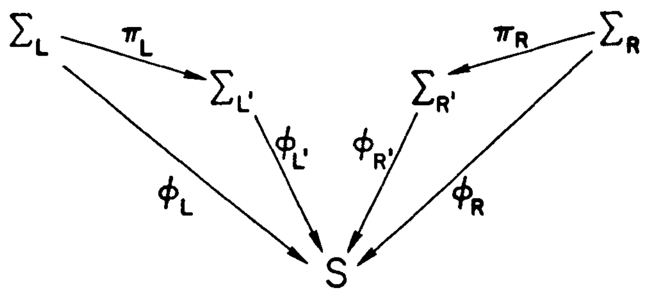
OBSERVATION 8. If g is recurrent, then any word whose product is g is magic for ϕ_R and ϕ_L.

PROOF. Let $a^1 \cdots a^n$ be a word so that the product $g = a^1 \cdots a^n$ is recurrent and examine ϕ_R. Suppose there are s symmetries. Then as h ranges through the elements of a principal component there will be exactly s distinct products hg. This means $a^1 \cdots a^n$ is magic. □

OBSERVATION 9. A sofic system is of finite type if and only if it can be described by pair $(\mathcal{G}, \mathcal{A})$ where every element is recurrent.

PROOF. In Section 1 we showed how to describe any ssft by a pair $(\mathcal{G}, \mathcal{A})$. The semi-group produced by this construction contains only recurrent elements. Conversely, suppose the pair $(\mathcal{G}, \mathcal{A})$ is such that every element is recurrent. Then every word is magic for ϕ_R. In particular each $a \in \mathcal{A}$ is magic. Letting s be the number of symmetries, there are always the same s endpoints to any path ending in a. Each of these endpoints (group elements) has exactly the same successors as any other since they are in the same $\sim$ class. This means the successors for any occurrence of a are independent of the predecessors. □

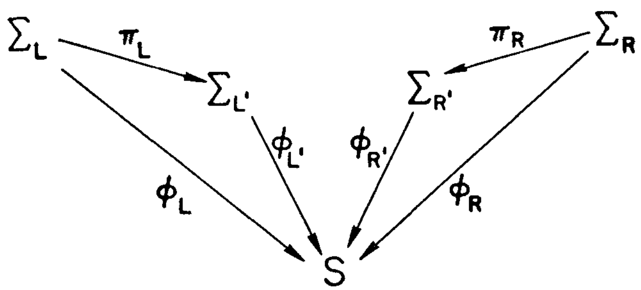
In several of the constructions that follow we will make use of a special type of semi-group homomorphism. If $(\mathcal{G}, \mathcal{A})$ and $(\mathcal{G}', \mathcal{A})$ are sofic pairs and $\pi : \mathcal{G} \to \mathcal{G}'$ is a semi-group homomorphism that is the identity on $\mathcal{A}$ and has $\pi^{-1}(0) = \{0\}$ then we will say it is a *homomorphism of sofic pairs* and we will denote it by $\pi : (\mathcal{G}, \mathcal{A}) \to (\mathcal{G}', \mathcal{A})$. This means the two pairs describe the same sofic system and that we have the following commutative diagram:



where (Σ_L, ϕ_L) and (Σ_R, ϕ_R) are the covers for $(\mathcal{G}, \mathcal{A})$, $(\Sigma_{L'}, \phi_{L'})$ and $(\Sigma_{R'}, \phi_{R'})$ are the covers for $(\mathcal{G}', \mathcal{A})$, and π_L and π_R are 1-block factor maps defined by restricting π to recurrent elements in $\mathcal{G}$. It is this parallel between homomorphisms of sofic pairs and factor maps between their covers that we will exploit in what follows.

Suppose $(\mathcal{G}, \mathcal{A})$ describes S and has $s > 1$ symmetries. We will produce another semi-group with the same generators that describe S. This new pair $(\mathcal{G}', \mathcal{A})$ will have only the identity symmetry. The idea is to simply identify all elements in an $\sim$ equivalence class. We must make sure that no generators are identified. Assume, for the moment, that no generators are recurrent. $\mathcal{G}' = \mathcal{G} \setminus \sim$ is a well-defined semi-group generated by $\mathcal{A}$. If $g \sim h$ and $gk, hk \neq 0$, then they are in the same R component, because they are in the same R component as g. They are in the same L component because multiplication by k is a translation of the L component containing g and h. This means $gk \sim hk$ and similarly $kg \sim kh$ when the products are nonzero. Consequently if $g \sim g'$ and $h \sim h'$, then $gh \sim g'h \sim g'h'$. Clearly $\mathcal{A}$ generates $\mathcal{G}'$ and the projection $\pi : \mathcal{G} \to \mathcal{G}'$ is a homomorphism of sofic pairs. Except for the assumption on the generators we have proven:

THEOREM 5. *If* $(\mathcal{G}, \mathcal{A})$ *defines* S *and has* $s > 1$ *symmetries then* S *may be described by a sofic pair* $(\mathcal{G}', \mathcal{A})$ *which has only one symmetry and gives a commutative diagram*:



where the factor maps π_L *and* π_R *are exactly* s-*to*-1.

COROLLARY [6], [4]. *Every irreducible sofic system is the* 1-*to*-1 *a.e. image of a subshift of finite type.*

The last step is to show that we can change $\mathcal{G}$ without changing (Σ_L, ϕ_L) and (Σ_R, ϕ_R) so that in the new semi-group $\bar{\mathcal{G}}$, all the generators are transient. Begin with $(\mathcal{G}, \mathcal{A})$, for each recurrent $a \in \mathcal{A}$ add an element $\bar{a}$ to the set $\mathcal{G}$. The result is $\bar{\mathcal{G}}$. If $a^1 \cdots a^n$ is a word in $\mathcal{A}^n$, $a^1 \cdots a^n = a$ in $\mathcal{G}$ and $a^1 \cdots a^n$ is not the word consisting of the generator a alone, define $a^1 \cdots a^n = \bar{a}$ in $\bar{\mathcal{G}}$. Leave all other elements in $\mathcal{G}$ unchanged. There is a natural homomorphism $\pi : (\bar{\mathcal{G}}, \mathcal{A}) \rightarrow (\mathcal{G}, \mathcal{A})$. Notice that all generators are transient in $\bar{\mathcal{G}}$ and that the covers (Σ_L, ϕ_L) and (Σ_R, ϕ_R) are unchanged. To see that some such construction is necessary consider the two-shift defined by $\mathcal{G} = \{a, b\}$, $\mathcal{A} = \{a, b\}$, with relations $ab = ba = a$ and $aa = bb = b$.

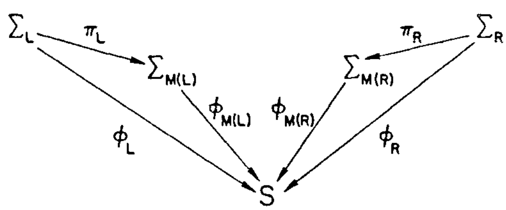
5. Minimal covers

A sofic system S has two canonical covers associated to it [6], [10]. To define them let $(\mathcal{G}, \mathcal{A})$ be a sofic pair describing S and begin by considering $\mathcal{W}(S)$. Define an equivalence relation $\sim$ on this set by saying $u \sim v$ if and only if $f(u) = f(v)$. There are finitely many equivalence classes. We are going to define a ssft whose alphabet is the set of equivalence classes. For two equivalence classes $[u]$ and $[v]$, let $A'([u], [v])$ be the number of generators $a \in \mathcal{A}$ such that $ua \in [v]$. Notice this is independent of the choice of $u \in [u]$. We now have a cover (Σ_A, ϕ) for S, where A is the unique maximal component of A', ϕ is defined by the edge labelling of A and is right resolving. From now on we will denote this cover by $(\Sigma_{M(R)}, \phi_{M(R)})$ and refer to it as the *minimal right cover on the symbol set $\mathcal{A}$*. The *minimal left cover* $(\Sigma_{M(L)}, \phi_{M(L)})$ is defined in the same manner but by using predecessors instead of successors. These were first defined in [6].

OBSERVATION 10. $(\Sigma_{M(R)}, \phi_{M(R)})$ and $(\Sigma_{M(L)}, \phi_{M(L)})$ are identical for any semi-groups defining the same sofic system on the same generators.

PROOF. The equivalence classes for words are defined without reference to the semi-group itself. The only thing that matters is when a product is nonzero. For a fixed set of generators this is independent of which semi-group is chosen to describe S. □

OBSERVATION 11. Suppose (Σ_R, ϕ_R) and (Σ_L, ϕ_L) are the covers of S that arise from a semi-group description of S. Then we have the following commutative diagram:



of one block factor maps.

PROOF. Suppose Σ_R is defined by the graph of a principal component of R, for some sofic pair $(\mathcal{G}, \mathcal{A})$. Define a factor map $\pi_R : \Sigma_R \to \Sigma_{M(R)}$ by identifying any two elements in this component with the same successors. □

6. 1-to-1 a.e. covers

In this section we will fix a sofic system S and consider its 1-to-1 a.e. covers. We will see that all such covers arise from semi-groups and that it is possible to associate to each cover or pair of covers a certain natural minimal semi-group.

OBSERVATION 12. If (Σ_A, ϕ) is a 1-to-1 a.e. right resolving cover for S then there is a sofic pair $(\mathcal{G}, \mathcal{A})$ that describes S and has (Σ_A, ϕ) as its right cover.

PROOF. Fix S on generators $\mathcal{A}$ and a 1-to-1 a.e. right resolving cover (Σ_A, ϕ). Associate to each word $a^1 \cdots a^n \in \mathcal{W}$ a subset $s(a^1 \cdots a^n)$ of $\mathcal{P}(V_A \times V_A)$ defined by

$$s(a^1 \cdots a^n) = \{(i,j): \text{path } i \to i_1 \to i_2 \to \cdots \to i_{n-1} \to j \text{ in the graph of } A \text{ that maps to } a^1 \cdots a^n\}.$$

Then define an equivalence relation on $\mathcal{W}$ by putting $a^1 \cdots a^n \sim b^1 \cdots b^m$ if and only if $s(a^1 \cdots a^n) = s(b^1 \cdots b^m)$. Together with 0 the equivalence classes make up a finite semi-group $\mathcal{G}$, generated by $\mathcal{A}$, where multiplication is defined by

$$[a^1 \cdots a^n][b^1 \cdots b^m] = \begin{cases} [a^1 \cdots a^n b^1 \cdots b^m] & \text{if } a^1 \cdots a^n b^1 \cdots b^n \in \mathcal{W}, \\ 0 & \text{otherwise.} \end{cases}$$

At this point we must deal with a special case. If $s(a) \neq s(b)$ for any $a, b \in \mathcal{A}$ we proceed as stated. If $s(a) = s(b)$ for some $a, b \in \mathcal{A}$ we must make a slight modification. We do what we did in the proof of Theorem 5. Define a and b to be the only words in their equivalence classes and treat all other words as already stated. This makes a and b transient elements but doesn't change (Σ_A, ϕ). In the rest of this paper we will always assume that all semi-groups have this property, namely if $s(a) = s(b)$ in either (Σ_L, ϕ_L) or (Σ_R, ϕ_R) then a and b are words such that no other words are equivalent to them in $\mathcal{G}$.

The equivalence class of any magic word is a recurrent element for $\mathcal{G}$. Since ϕ is 1-to-1 a.e. $a^1 \cdots a^n$ is magic if and only if $s(a^1 \cdots a^n) = \{(i_1, j), \ldots, (i_r, j)\}$ for some $j \in V_A$. This implies that the right cover for $(\mathcal{G}, \mathcal{A})$ is (Σ_A, ϕ). □

Definition. Given (Σ_A, ϕ) a right resolving 1-to-1 a.e. cover for S we say that the pair produced by the above construction is the *minimal semi-group for* (Σ_A, ϕ).

The reason for singling out this particular semi-group and calling it minimal will soon become clear. First we must examine the left cover (Σ_L, ϕ_L) that is produced.

Observation 13. If $(\mathcal{G}, \mathcal{A})$ is the minimal semi-group for (Σ_A, ϕ), a right resolving 1-to-1 a.e. cover for S, then (Σ_L, ϕ_L) is the minimal left cover for S on the symbols A.

Proof. Let $\mathcal{C}$ be a principal left component for $(\mathcal{G}, \mathcal{A})$. By Observation 11 we need only to see that if $g, h \in \mathcal{C}$ then $p(g) \neq p(h)$. A recurrent element of $\mathcal{C}$ has $s(a^1 \cdots a^n) = \{(i_1, j), \ldots, (i_r, j)\}$. By the duality that exists between R and L we know that a principal L component is made up of all recurrent elements, $b^1 \cdots b^m$, that have each member of $s(b^1 \cdots b^m)$ ending in j, a fixed element of V_A. Suppose $g = [a^1 \cdots a^n]$ and $h = [b^1 \cdots b^m]$ have the same predecessors,

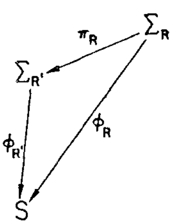
$$s(a^1 \cdots a^n) = \{(i_1, j), \ldots, (i_r, j)\} \quad \text{and} \quad s(b^1 \cdots b^m) = \{(i'_1, j), \ldots, (i'_t, j)\}.$$

For each $(i_k, j) \in s(a^1 \cdots a^n)$ there is a magic word $c_k^1 \cdots c_k^l$ such that

$s(c_k^1 \cdots c_k^l) = \{(i_1'', i_k), \ldots, (i_s'', i_k)\}$. So $s(c_k^1 \cdots c_k^l a^1 \cdots a^n) = \{(i_1'', j), \ldots, (i_s'', j)\}$ and $c_k^1 \cdots c_k^l \in p(g) = p(h)$. Since $c_k^1 \cdots c_k^l \in p(h)$ picks out i_k, one of the symbols $\{i_1', \ldots, i_t'\}$ is i_k. This says $s(a^1 \cdots a^n) \subseteq s(b^1 \cdots b^m)$. Since the argument is symmetric they are equal. □

Recall that in general if a word is recurrent then it is magic. Notice in Observation 13 that a word from $(\mathscr{G}, \mathscr{A})$ is recurrent if and only if it is magic for ϕ.

Observation 14. Let $(\Sigma_{R'}, \phi_{R'})$ be a 1-to-1 a.e. right resolving cover of S and let $(\mathscr{G}', \mathscr{A})$ denote the minimal semi-group for this cover. Let $(\mathscr{G}, \mathscr{A})$ be another sofic pair that describes S, and let (Σ_R, ϕ_R) be its right cover. Suppose we have a 1-block factor map $\pi_R : \Sigma_R \to \Sigma_{R'}$ such that the diagram



commutes. Then there exists a homomorphism $\phi : (\mathscr{G}, \mathscr{A}) \to (\mathscr{G}', \mathscr{A})$ of sofic pairs that induces this diagram.

Proof. Write $g \in \mathscr{G}$ as a product of generators $g = a^1 \cdots a^n$ and define $\pi(g)$ to be the product of $a^1 \cdots a^n$ in $\mathscr{G}'$. To see that π is well-defined let $g = a^1 \cdots a^n = b^1 \cdots b^m$ in $\mathscr{G}$. Fix a principal component $\mathscr{C}$ of R and consider $ha^1 \cdots a^n = hb^1 \cdots b^m$ for $h \in \mathscr{C}$. Applying π_R to these and all other such nonzero blocks we find that $s(a^1 \cdots a^n) = s(b^1 \cdots b^m)$ in $\Sigma_{R'}$. □

As a special case we have the following:

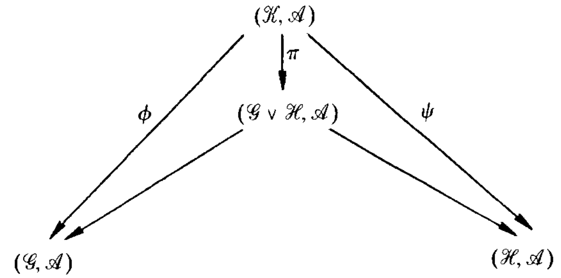
Observation 15. Suppose $(\mathscr{G}, \mathscr{A})$ describes S and has one symmetry. If $(\mathscr{G}', \mathscr{A})$ is the minimal semi-group for (Σ_R, ϕ_R), the right cover for $(\mathscr{G}, \mathscr{A})$, then there exists a homomorphism $\pi : (\mathscr{G}, \mathscr{A}) \to (\mathscr{G}', \mathscr{A})$ of sofic pairs.

We define the product of two sofic pairs $(\mathscr{G}, \mathscr{A})$ and $(\mathscr{H}, \mathscr{A})$ that describe the same sofic system. Let $\mathscr{G} \vee \mathscr{H}$ consist of all pairs $(g, h) \in \mathscr{G} \times \mathscr{H}$ such that there exists a word with its product $a^1 \cdots a^n = g$ in $\mathscr{G}$ and $a^1 \cdots a^n = h$ in $\mathscr{H}$. Define the multiplication in $\mathscr{G} \vee \mathscr{H}$ by

$$(g, h)(g', h') = \begin{cases} (gg', hh') & \text{if } gg' \neq 0 \text{ in } \mathscr{G}, \\ 0 & \text{otherwise.} \end{cases}$$

Identifying $a \in \mathscr{A}$ with $(a, a) \in \mathscr{G} \vee \mathscr{H}$ we obtain the sofic pair $(\mathscr{G} \vee \mathscr{H}, \mathscr{A})$ that describes the same sofic system as does $(\mathscr{G}, \mathscr{A})$. There are natural homomorphisms of sofic pairs from $(\mathscr{G} \vee \mathscr{H}, \mathscr{A})$ onto $(\mathscr{G}, \mathscr{A})$ and $(\mathscr{H}, \mathscr{A})$. This product has several easily verified properties.

Observation 16. If $(\mathscr{K}, \mathscr{A})$ is a sofic pair and there are homomorphisms $\phi : (\mathscr{K}, \mathscr{A}) \to (\mathscr{G}, \mathscr{A})$ and $\psi : (\mathscr{K}, \mathscr{A}) \to (\mathscr{H}, \mathscr{A})$ of sofic pairs then there is a homomorphism $\pi : (\mathscr{K}, \mathscr{A}) \to (\mathscr{G} \vee \mathscr{H}, \mathscr{A})$ of sofic pairs that makes the following diagram commute:



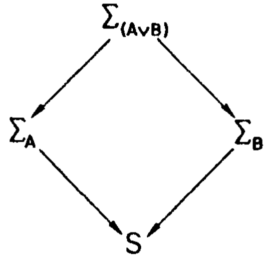
Observation 17. An element $(g, h) \in \mathscr{G} \vee \mathscr{H}$ is recurrent if and only if g is recurrent in $\mathscr{G}$ and h is recurrent in $\mathscr{H}$.

If (Σ_A, ϕ) and (Σ_B, ψ) are covers for a sofic system S we can define the *fibered product over ϕ and ψ* (this is a standard construction, see for example [1]). If ϕ and ψ are 1-block maps then define Σ_C by:

$$L_C = \{(i, j) : i \in L_A,\ j \in L_B \text{ and } \phi(i) = \psi(j)\}$$

and say $(i, j) \to (i', j')$ if $i \to i'$ in Σ_A and $j \to j'$ in Σ_B.

There are natural factor maps $\psi' : \Sigma_C \to \Sigma_A$ and $\phi' : \Sigma_C \to \Sigma_B$ defined by $\psi'(i, j) = i$ and $\phi'(i, j) = j$. The ssft Σ_C may not be irreducible but it will contain an inrreducible component $\Sigma_{(A \vee B)}$, such that the restrictions of the maps ψ' and ϕ' to it are still onto Σ_A and Σ_B. This results in the following commutative diagram:



and $\Sigma_{(A\vee B)}$ is the minimal cover of S that gives such a diagram.

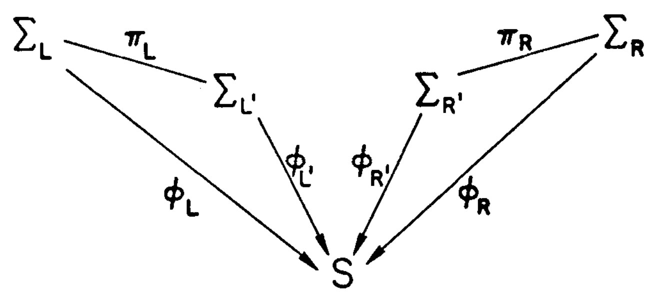
OBSERVATION 18. Let $(\mathcal{G}_1, \mathcal{A})$ and $(\mathcal{G}_2, \mathcal{A})$ be sofic pairs that describe the same system S. Let $(\Sigma_{L_1}, \varphi_{L_1})$ and $(\Sigma_{R_1}, \varphi_{R_1})$ be the covers for $(\mathcal{G}_1, \mathcal{A})$ and $(\Sigma_{L_2}, \varphi_{L_2})$ and $(\Sigma_{R_2}, \varphi_{R_2})$ be the covers for $(\mathcal{G}_2, \mathcal{A})$. Then $(\mathcal{G}_1 \vee \mathcal{G}_2, \mathcal{A})$ has $(\Sigma_{L_1\vee L_2}, \varphi_{L_2}\circ\varphi'_{L_1})$ for its left cover and $(\Sigma_{(R_1\vee R_2}, \varphi_{R_2}\circ\varphi'_{R_1})$ for its right cover.

THEOREM 6. *Let (Σ_A, ϕ_A) be a 1-to-1 a.e. left resolving cover of S, and let (Σ_B, ϕ_B) be a 1-to-1 a.e. right resolving cover. The system S may be described by a sofic pair $(\mathcal{G}, \mathcal{A})$ which has (Σ_A, ϕ_A) and (Σ_B, ϕ_B) for its covers.*

PROOF. Let $(\mathcal{G}_1, \mathcal{A})$ be the minimal semi-group for (Σ_A, ϕ_A) and let $(\mathcal{G}_2, \mathcal{A})$ be the minimal one for (Σ_B, ϕ_B). By Observation 13, the left cover for $(\mathcal{G}_2, \mathcal{A})$ is the minimal cover $(\Sigma_{M(L)}, \phi_{M(L)})$ so that by Observation 18 the left cover for $(\mathcal{G}_1 \vee \mathcal{G}_2, \mathcal{A})$ is $\Sigma_{(A\vee M(L))}$. But Observation 11 implies that $\Sigma_{(A\vee M(L))} = \Sigma_A$. A similar argument shows that (Σ_B, ϕ_B) is the right cover for $(\mathcal{G}_1 \vee \mathcal{G}_2, \mathcal{A})$. □

We will now say that this group is the *minimal semi-group for* (Σ_A, ϕ) *and* (Σ_B, ψ). It is easy to see that minimal here means the same as before.

OBSERVATION 19. Let the pair $(\mathcal{G}', \mathcal{A})$ have one symmetry and be minimal for its covers $(\Sigma_{L'}, \phi_{L'})$ and $(\Sigma_{R'}, \phi_{R'})$. Let $(\mathcal{G}, \mathcal{A})$ be a pair that describes the same sofic system S, and let (Σ_R, ϕ_R) and (Σ_L, ϕ_L) denote the covers for $(\mathcal{G}, \mathcal{A})$. Suppose that we have 1-block factor maps $\pi_L : \Sigma_L \to \Sigma_{L'}$ and $\pi_R : \Sigma_R \to \Sigma_{R'}$ such that the diagram



commutes. Then there exists a homomorphism $\pi : (\mathcal{G}, \mathcal{A}) \to (\mathcal{G}', \mathcal{A})$ of sofic pairs such that the above commutative diagram is the one obtained from π.

PROOF. Use the proof of Theorem 6 and Observations 14 and 16. □

As a special case of the above observation we have

OBSERVATION 20. Let $(\mathcal{G}, \mathcal{A})$ be a sofic pair that has one symmetry. If $\mathcal{G}'$ is the minimal semi-group for its left and right covers, then there exists a homomorphism $\pi : (\mathcal{G}, \mathcal{A}) \to (\mathcal{G}', \mathcal{A})$ of sofic pairs.

In particular, we now know that the system S may be described by a sofic pair that has $(\Sigma_{M(L)}, \phi_{M(L)})$ and $(\Sigma_{M(R)}, \phi_{M(R)})$ for its covers and is minimal for these covers. This pair will be called the *minimal pair for S (on the symbols $\mathcal{A}$)*.

OBSERVATION 21. If $(\mathcal{G}', \mathcal{A})$ is the minimal sofic pair for S and $(\mathcal{G}, \mathcal{A})$ is a sofic pair that also describes S then there exists a homomorphism $\pi : (\mathcal{G}, \mathcal{A}) \to (\mathcal{G}', \mathcal{A})$.

PROOF. Use Observations 11 and 19.

We end the section with an intrinsic characterization of minimal sofic pairs.

OBSERVATION 22. A sofic pair $(\mathcal{G}, \mathcal{A})$ with one symmetry is minimal for its covers if and only if the following condition is met:

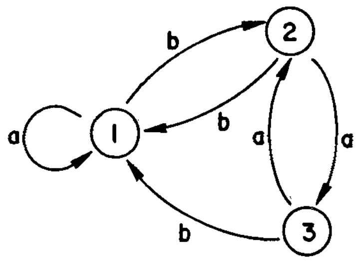
(*) If $g, h \in \mathcal{G}$ are such that $gk = hk$ and $kg = kh$ for all recurrent $k \in \mathcal{G}$, then $g = h$.

PROOF. If (*) fails, identify any two elements $g, h \in \mathcal{G}$ such that $gk = hk$ and $kg = kh$ for all recurrent $k \in \mathcal{G}$ to obtain a new sofic pair $(\mathcal{G}', \mathcal{A})$ and a homomorphism $\pi : (\mathcal{G}, \mathcal{A}) \to (\mathcal{G}', \mathcal{A})$. Observe that ψ is bijective when restricted to the recurrent elements of $\mathcal{G}$ to conclude that $(\mathcal{G}', \mathcal{A})$ has the same covers as $(\mathcal{G}, \mathcal{A})$ and the latter pair cannot be minimal for its covers.

To prove the converse, suppose (*) holds. Let $(\mathcal{G}', \mathcal{A})$ be the minimal pair for the covers of $(\mathcal{G}, \mathcal{A})$ and let $\psi : (\mathcal{G}, \mathcal{A}) \to (\mathcal{G}', \mathcal{A})$ be a homomorphism. We will see that ψ is an isomorphism. First note that, since the two sofic pairs have the same covers, ψ is injective when restricted to the recurrent elements of $\mathcal{G}$. Suppose $g, h \in \mathcal{G}$ are such that $\psi(g) = \psi(h)$, and let k be a recurrent element of $\mathcal{G}$. Since $\psi(g)\psi(k) = \psi(h)\psi(k)$, either the products gk and hk are both zero or they are recurrent elements of $\mathcal{G}$ which have the same image under ψ; in the latter case we have $gk = hk$ since ψ does not identify distinct recurrent elements. Thus, $gk = hk$ for all recurrent k. Similarly, $kg = kh$ for all recurrent k, and (*) then gives that $g = h$. □

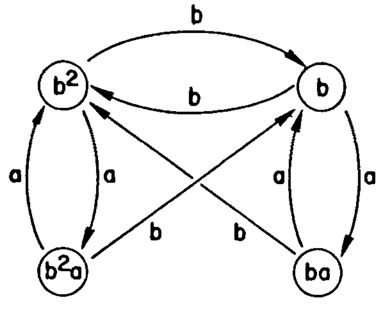
7. *k*-to-1 a.e. covers

Let us begin with an example to see how the situation changes when we are dealing with k-to-1 a.e. maps. This example was constructed with the help of Brian Marcus. Define (Σ_A, ϕ) a 2-to-1 a.e. cover of the full 2-shift by



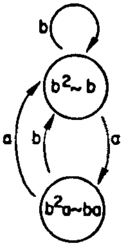
Notice that ϕ is 3-to-1 in some places. It cannot be the right cover for any semi-group because, if it were, then by Theorem 5 if would collapse $\Sigma_A \to \Sigma_B \to \Sigma_2$, $\phi = \psi \circ \pi$, where π would be 2-to-1 everywhere, ψ 1-to-1 a.e., and so $\phi = \psi \circ \pi$ would have to be $2k$-to-1, for some k, on each point.

We can construct a semi-group to get



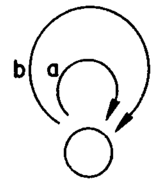
as its right cover (Σ_R, ϕ_R).

Notice this map is 2-to-1 a.e. but 4-to-1 in some places. Collapsing by the symmetry gives a semi-group with

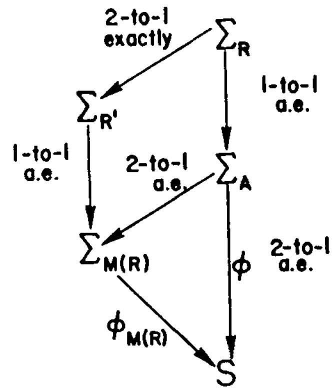


as its right cover $(\Sigma_{R'}, \varphi_{R'})$.

This factors onto the minimal right cover by a 1-to-1 a.e. map.



The picture we have is

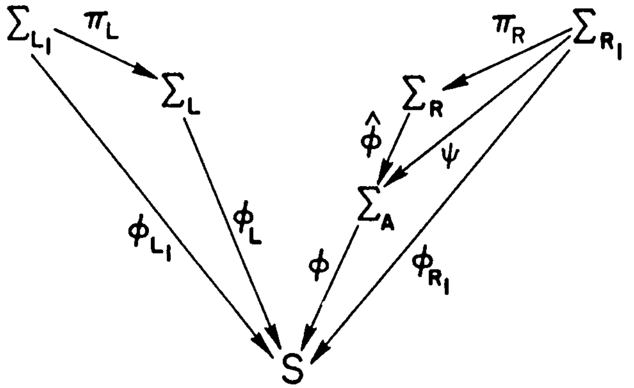


This is the general situation. The rest of this section consists of a series of observations about k-to-1 a.e. covers for sofic systems that are analogous to the ones previously made about 1-to-1 a.e. covers. No proofs of these observations are included.

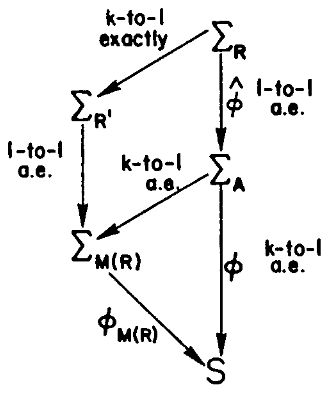
If (Σ_A, ϕ) is a k-to-1 a.e. right resolving cover for S then it is possible to construct a sofic pair $(\mathcal{G}, \mathcal{A})$ that we will call the *minimal sofic pair for* (Σ_A, ϕ).

OBSERVATION 23. Let (Σ_A, ϕ) be a k-to-1 a.e. right resolving cover for S, let $(\mathcal{G}, \mathcal{A})$ be its minimal sofic pair and let (Σ_R, ϕ_R) be the right cover for this pair. Then there is a 1-to-1 a.e. right resolving factor map $\hat{\phi} : \Sigma_R \to \Sigma_A$ such that $\phi_R = \phi \circ \hat{\phi}$.

OBSERVATION 24. Let (Σ_A, ϕ) be a k-to-1 a.e. right resolving cover for S, let $(\mathcal{G}, \mathcal{A})$ be its minimal sofic pair with (Σ_R, ϕ_R) as its right cover. If $(\mathcal{G}_1, \mathcal{A})$ is another sofic pair that describes S, $(\Sigma_{R_1}, \phi_{R_1})$ is its right cover and there is a 1-block factor map $\psi : \Sigma_{R_1} \to \Sigma_A$ such that $\phi_{R_1} = \phi \circ \psi$ then there is a homomorphism $\pi : (\mathcal{G}_1, \mathcal{A}) \to (\mathcal{G}, \mathcal{A})$ of sofic pairs and we have the following commutative diagram:



THEOREM 7. *Lét* (Σ_A, ϕ) *be a* k*-to-*1 *a.e. cover for* S *and let* $(\mathscr{G}, \mathscr{A})$ *be its minimal sofic pair. Then we have the following commutative diagram*:



Here (Σ_R, ϕ_R) *is the right cover of* $(\mathscr{G}, \mathscr{A})$, $(\Sigma_{R'}, \phi_{R'})$ *is the right cover for the sofic pair* $(\mathscr{G}', \mathscr{A})$ *obtained by collapsing the symmetries of* $(\mathscr{G}, \mathscr{A})$, *and* $(\Sigma_{M(R)}, \phi_{M(R)})$ *is the minimal right cover for* S.

COROLLARY [2]. *If* (Σ_A, ϕ) *is a* k*-to-*1 *a.e. right resolving cover for* S *then* ϕ *factors through the minimal right cover for* S.

If (Σ_A, ϕ) is a k-to-1 a.e. left resolving cover for S and (Σ_B, ϕ_B) is a k-to-1 a.e. right resolving for S then as in the 1-to-1 a.e. case it is possible to define the *minimal sofic pair for* (Σ_A, ϕ_A) and (Σ_B, ϕ_B). This pair has all the minimality conditions that one would expect.

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